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Schild's Ladder Parallel Transport Procedure for an Arbitrary Connection

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We analyze the Schild's ladder parallel transport procedure for an arbitrary connection. We demonstrate that the procedure, while it can be performed for any connection, in fact is only capable of detecting the symmetric part of this connection. In geometries with a symmetric connection it fulfills its goal to express connection and parallel transport of any vector in terms of geodesics of such geometries.

1. INTRODUCTION

The Schild's ladder parallel transport procedure [2, 4] is known to provide a strikingly simple depiction of the parallel transport of a vector along a curve. It is even more remarkable that it utilizes the tangent structure of a manifold only for the initial formulation of the problem and for interpretation of the final results. The technical part of the procedure is based entirely on the manifold itself and requires only the knowledge of geodesics, thus implying that all information about the connection of the geometry is encoded in the geodesics.

The geodesic feature makes the procedure a promising tool for extensions of the connection concept to the formulations of general relativity lacking differentiability features and the tangent structures, such as various lattice formulations, including the most promising Regge calculus formulation [3].

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A careful analysis of the Schild's ladder procedure is absent in the literature. The "proof" that it works properly refers to one's intuition and to the equivalence principle, and essentially cannot be considered as a proof. At best, it is merely a supporting argument for it. One feature which calls for caution is the symmetry of the procedure in its treatment of the displacement vector and of the parallel translated vector. This symmetry serves as the basis of the "proof" of the connection symmetry in ref. 4, while the procedure itself does not contain any references to the connection symmetry. It can be carried out for the connections that are not symmetric, leaving one with the question of what it will do in this case.

The present paper attempts to close the gap by figuring out, completely and in the most general context, what the Schild's ladder procedure is detecting. This is achieved via a careful formulation of the procedure with particular stress on the structures on which different elements of the construction reside (Section 2). After that, a simple calculation shows what exactly the procedure involves (Section 3).

2. SCHILD'S LADDER PROCEDURE

Our description of the Schild's ladder procedure mirrors the description given in ref. 4, except we are more careful and explicit in pointing out where different elements of the Schild's construction reside. We assume that the geometry on the manifold M is determined by an arbitrary connection ∇. Connections on the curves in the manifold are induced by this connection in the usual manner [1]. Knowledge of these induced connections along all the curves in the manifold is equivalent to the knowledge of the manifold connection. The Schild's ladder procedure supplies a prescription for recovering these connections.

The procedure considers a curve C parametrized by the parameter τ and a vector $\mathbf{A} \in T_{P_0}M$ at a point P_0 on the curve ($T_{P_0}M$ is the tangent space at the point P_0) (cf. Fig. 1). The direction of the curve C at P_0 is given by the vector $\mathbf{u}$ at this point ($\mathbf{u} \in T_{P_0}M$).

The Schild's ladder algorithm considers the parallel transport of vector $\mathbf{A}$ from the point $P_0 \in C$ to the point $P_1 \in C$ separated from P_0 by the parameter value τ (presumably infinitesimal) and works as follows (cf. Fig. 1).

1. We pick a curve in the manifold passing through the point P_0 in the direction of $\mathbf{A}$ and parametrized by a parameter σ in such a way that, at P_0, $d/d\sigma = \mathbf{A}$. After that, we pick a point P_2 on this curve separated from the point P_0 by the (infinitesimal) value of the parameter σ. Both τ and σ can be thought to be small enough for the whole Schild's ladder construction to be placed within one coordinate neighborhood. All coordinate expressions

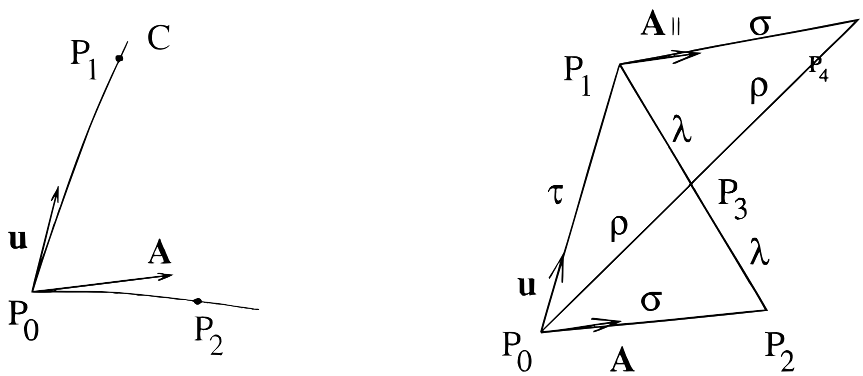

Fig. 1. Vector **A** is to be parallel transported along the curve C from point P_0 to point P_1. The sections of the curves involved in the Schild's ladder procedure are infinitesimal and thus are depicted as straight on all subsequent figures. The actual Schild's ladder parallel transport procedure is shown on the right in this figure.

will be written in the chart of this coordinate neighborhood, and, up to the order relevant in subsequent computations (linear in τ and linear in σ)

$$x_1^\mu - x_0^\mu = \tau u^\mu \tag{1}$$

$$x_2^\mu - x_0^\mu = \sigma A^\mu \tag{2}$$

The lower index in these equations labels the point, while u^μ, A^μ are the components of vectors in the coordinate basis (index labeling of the point is omitted for these components as there is no possibility of confusion).

2. The action of the second step of the procedure is performed entirely in the manifold. Points P_2 and P_1 are connected by a geodesic parametrized by the affine parameter λ in such a way that the point P_2 corresponds to the zero value of this parameter and P_1 corresponds to the value of the parameter 2λ. Point P_3 on this geodesic is "the middle point" determined by the value of the parameter λ. After that, point P_0 is connected to the point P_3 by the geodesic parametrized by the parameter ρ in such a way that this parameter takes zero value at P_0 and value ρ at P_3. The geodesic is continued past the point P_3 until the point P_4 determined by the value of the parameter 2ρ is reached. A curve connecting points P_1 and P_4 is parametrized in such a way that P_1 corresponds to the zero value of the parameter and P_4 corresponds to the parameter value σ. This curve might be thought of as the curve connecting P_0 and P_1 parallel translated from P_0 to P_1 along the curve C.

3. The vector $\mathbf{A}_\parallel = (d/d\sigma)\,\mathrm{P}_1$ is thought of as the vector **A** parallel translated along C from P_0 to P_1. Just as in the first step of the procedure, the value of σ is considered to be infinitesimal, so that, to the relevant order,

$$x_4^\mu - x_1^\mu = \sigma \mathbf{A}_\parallel^\mu \tag{3}$$

and

$$(\sigma \mathbf{A})_\parallel = \sigma \mathbf{A}_\parallel \tag{4}$$

The procedure might seem to be slightly ambiguous in the part concerning the curves (and their parametrizations) connecting P_0 to P_2 and P_1 to P_4. In fact, there is no ambiguity, as the whole construction is infinitesimal and thus all the relations (between vectors and displacements, between different parameters, etc.) are linearized. In possible generalizations that do not appeal to infinitesimal properties of curves P_0P_2 and P_1P_4, the ambiguity can be removed by imposing on these curves the demand to be geodesics with the affine parameter σ chosen the way described above. In infinitesimal limit, the two descriptions coincide.

This, in essence, is the reason why simple diagrams similar to the one in ref. 4 work, to an extent. However, the "proof" that they work given in ref. 4 is a bit naive and leads to several questions. The procedure itself does not appear to have any restrictive assumptions concerning the connection. It could be carried out for the geometries determined by an arbitrary connection.

On the other hand, the procedure essentially states that parallel translation of any vector and, consequently, all of the connection is determined by geodesics of the geometry. This is strange because a geodesic with the affine parameter λ and its tangent vector $\mathbf{u} = d/d\lambda$ with components u^μ in a coordinate basis is described by the equation

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta}\, u^\alpha u^\beta = 0 \tag{5}$$

which clearly cannot provide any information about the antisymmetric part of the connection. The geodesics determine only the symmetric part of the connection. As a result, the Schild's ladder procedure can possibly work only for symmetric connections. This conclusion is dramatically stressed by the "proof" that the connection must be symmetric [4] based on the observation that the Schild's ladder diagram above is symmetric with respect to the displacement vectors $\sigma\mathbf{A}$ and $\tau\mathbf{u}$ (cf. Fig. 2) and allows one, infinitesimally, to interpret $\tau\mathbf{u}_\parallel$ as the displacement from P_2 to P_4,

$$x_4^\mu - x_2^\mu = \tau u_\parallel^\mu \tag{6}$$

This makes the diagram formed by the vectors $\sigma\mathbf{A}$, $\tau\mathbf{u}_\parallel$ and $\tau\mathbf{u}$, $\sigma\mathbf{A}_\parallel$ closed, which implies [4]

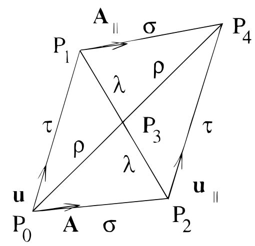

Fig. 2. The Schild's ladder parallel transport procedure is symmetric with respect to the displacement vectors σ **A** and τ**u**.

$$\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\beta\alpha} \tag{7}$$

3. SCHILD'S LADDER AND PARALLEL TRANSPORT

To trace the origin of this symmetry, we compute directly the difference $A_{\|}^{\mu} - A^{\mu}$ from the infinitesimal picture in a local coordinate system, similar to the one above, except we augment the picture with additional details describing geodesics connecting points P_0 to P_4 and P_1 to P_2 (cf. Fig. 3).

We introduce notations $\mathbf{v}$ and $\mathbf{v}_{\|}$ for the vector $d/d\rho$ at the points P_0 and P_3, respectively, as well as $\mathbf{w}$ and $\mathbf{w}_{\|}$ for the vector $d/d\lambda$ at the points P_2 and P_3. Vector $\mathbf{v}_{\|}$ is obtained by the parallel translation of $\mathbf{v}$ along the geodesic connecting P_0 to P_4.

The equation for the geodesic

$$\frac{dv^{\mu}}{d\rho} + \Gamma^{\mu}{}_{\alpha\beta} v^{\alpha} \frac{dx^{\beta}}{d\rho} = 0 \tag{8}$$

produces, in the lowest order, the relation

$$v_{\|}^{\mu} - v^{\mu} = -(\Gamma^{\mu}{}_{\alpha\beta})_0 \, v^{\alpha}(x_3^{\beta} - x_0^{\beta}) \tag{9}$$

where the index 0 on the connection coefficients indicates that they are computed at the point P_0. This index, indicating the point of computing the

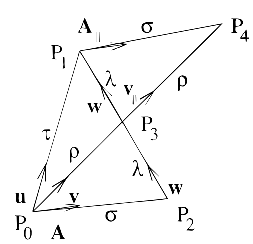

Fig. 3. Computation of the results of Schild's ladder transport, demonstrating that Schild's procedure picks up the symmetric part of the connection.

connection coefficients, can be dropped because the change of this point on the diagram results in corrections of higher order. We will do so from now on.

The quantity of interest for our computation is

$$\rho v_{\|}^{\mu} - \rho v^{\mu} = -\rho\Gamma^{\mu}{}_{\alpha\beta}\, v^{\alpha}(x_3^{\beta} - x_0^{\beta}) \tag{10}$$

The same line of consideration produces a similar equation for **w**, $\mathbf{w}_{\|}$:

$$\lambda w_{\|}^{\mu} - \lambda w^{\mu} = -\lambda\Gamma^{\mu}{}_{\alpha\beta}\, w^{\alpha}(x_3^{\beta} - x_2^{\beta}) \tag{11}$$

We are going to evaluate the difference between the displacements from point P_1 to point P_4 and the displacement from point P_0 to point P_2,

$$(x_4^{\mu} - x_1^{\mu}) - (x_2^{\mu} - x_0^{\mu}) = (x_4^{\mu} - x_3^{\mu}) + (x_3^{\mu} - x_1^{\mu}) - (x_2^{\mu} - x_3^{\mu}) - (x_3^{\mu} - x_0^{\mu}) \tag{12}$$

In the lowest order

$$(x_4^{\mu} - x_1^{\mu}) = \sigma A_{\|}^{\mu} \tag{13}$$

$$(x_2^{\mu} - x_0^{\mu}) = \sigma A^{\mu} \tag{14}$$

$$(x_4^{\mu} - x_3^{\mu}) = \rho v_{\|}^{\mu} \tag{15}$$

$$(x_3^{\mu} - x_1^{\mu}) = -\lambda w_{\|}^{\mu} \tag{16}$$

$$(x_2^{\mu} - x_3^{\mu}) = -\lambda w^{\mu} \tag{17}$$

$$(x_3^{\mu} - x_0^{\mu}) = \rho v^{\mu} \tag{18}$$

Substitution of expressions (13)–(18) in (12) followed by application of Eqs. (10), (11) yields

$$\sigma(A_{\|}^{\mu} - A^{\mu}) = \lambda\Gamma^{\mu}{}_{\alpha\beta}\, w^{\alpha}(x_3^{\beta} - x_2^{\beta}) - \rho\Gamma^{\mu}{}_{\alpha\beta}\, v^{\alpha}(x_3^{\beta} - x_0^{\beta}) \tag{19}$$

In the order appropriate for substitution in Eq. (19) (cf. the diagram), we have

$$(x_3^{\beta} - x_2^{\beta}) = \tfrac{1}{2}(\tau u^{\beta} - \sigma A^{\beta}) \tag{20}$$

$$(x_3^{\beta} - x_0^{\beta}) = \tfrac{1}{2}(\tau u^{\beta} + \sigma A^{\beta}) \tag{21}$$

This substitution results in the equation

$$\begin{aligned}\sigma(A_{\|}^{\mu} - A^{\mu}) &= \tfrac{1}{2}\lambda\Gamma^{\mu}{}_{\alpha\beta}w^{\alpha}(\tau u^{\beta} - \sigma A^{\beta}) - \tfrac{1}{2}\rho\Gamma^{\mu}{}_{\alpha\beta}v^{\alpha}(\tau u^{\beta} + \sigma A^{\beta})\\ &= -\tfrac{1}{2}\tau\Gamma^{\mu}{}_{\alpha\beta}(-\lambda w^{\alpha} + \rho v^{\alpha})u^{\beta} - \tfrac{1}{2}\sigma\Gamma^{\mu}{}_{\alpha\beta}(\lambda w^{\alpha} + \rho v^{\alpha})A^{\beta}\end{aligned} \tag{22}$$

In the same order (cf. again the diagram)

$$\sigma A^{\alpha} = -\lambda w^{\alpha} + \rho v^{\alpha} \tag{23}$$

$$\tau u^{\beta} = \lambda w^{\alpha} + \rho v^{\alpha} \tag{24}$$

Substitution of the last two expressions in Eq. (22) leads to the final expressions

$$\sigma(A_{\|}^{\mu} - A^{\mu}) = -\tfrac{1}{2}\tau\sigma\Gamma^{\mu}{}_{\alpha\beta}(A^{\alpha}u^{\beta} + u^{\alpha}A^{\beta}) = -\tau\sigma\Gamma^{\mu}{}_{(\alpha\beta)}A^{\alpha}u^{\beta} \tag{25}$$

$$A_{\|}^{\mu} - A^{\mu} = -\tau\Gamma^{\mu}{}_{(\alpha\beta)}A^{\alpha}u^{\beta} \tag{26}$$

Equation (26) clearly states that the Schild's ladder procedure in the case of general connection determines parallel transport with respect to the symmetric part of this connection.

4. DISCUSSION

In Sections 2 and 3 we have provided a careful description of the Schild's parallel transport procedure clearly delineating the steps of the procedure performed on the manifold from the steps involving the differentiable (tangent) structure. A direct computation of Section 3 clearly demonstrates that, in the case of a generic connection, Schild's procedure does not describe parallel transport with respect to this connection and instead yields parallel transport with respect to the symmetric part of the connection. This result is natural in view of the fact that Schild's parallel transport procedure is based entirely on geodesic lines, and equations describing these lines are not capable of detecting anything but the symmetric part of the connection. If the original connection is symmetric (which is the case in such an important application as general relativity), Schild's procedure is sufficient. Moreover, the procedure can be split into parts some of which occur entirely in the manifold itself and do not appeal to the tangent spaces. This gives one hope that the procedure can be used in formulations of general relativity lacking tangent structure (Regge calculus, for instance) via slight reinterpretation of the parallel-transported objects.

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